

POWER OPERATIONS IN  $K$ -THEORY COMPLETED AT A PRIME

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ABSTRACT. We describe the action of power operations on the  $p$ -completed cooperation algebra  $K_0^\vee K = K_0(K)_p^\wedge$  for  $K$ -theory at a prime  $p$ .

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## INTRODUCTION

Power operations in suitably completed (co)homology theories have been studied and used by several authors, for example Rezk [24, 25]; the paper of Barthel and Frankland [11] provides a convenient account of this, in particular for the case of  $p$ -complete  $K$ -theory at a prime  $p$ .

In the present paper we describe the action of the  $\theta$ -operator (which we follow [11] in denoting by  $Q$ ) on the  $p$ -completed cooperation algebra

$$K_0^\vee K = K_0(K)_p^\wedge = \pi_*(L_{K(1)}(KU \wedge KU)).$$

We expect this to be of use in investigating the  $\theta$ -action and its interaction with the  $K_*^\vee K$ -coaction on  $K_*^\vee(A)$  for any  $E_\infty$  ring spectrum  $A$ . We also give some results on  $K_*^\vee(\mathbb{P}X)$ , where  $\mathbb{P}X$  denotes the free commutative  $S$ -algebra on a spectrum  $X$  introduced in [16].

It is likely that some of our results are known to experts, but we have not found a published source, so we feel it worthwhile writing them down. We also provide a brief appendix describing the relationship between continuous actions of the  $p$ -adic units and continuous coactions of  $K_0^\vee(K)$  on  $L$ -complete  $\mathbb{Z}_p$ -modules.

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An obvious related problem to investigate is that of describing the actions of power operations on  $K_0^\vee(BU)$  or equivalently on  $K_0^\vee(MU)$  (these actions correspond under the Thom isomorphism). The  $E_\infty$  orientation of [18] induces a morphism of  $\theta$ -algebras  $K_0^\vee(MU) \rightarrow K_0^\vee(K)$  but this is not injective on the image of the Hopf algebra primitives  $\text{Pr } K_0^\vee(BU)$ , and this seems to make the determination of the action on primitives more delicate than in the case of ordinary mod  $p$  homology as carried out by Kochman[19]. We plan to return to this in future work.

## 1. BACKGROUND ON COMPLETED $K$ -THEORY AND ITS POWER OPERATIONS

Throughout,  $p$  will be a prime and  $K = KU_{(p)}$  will denote the  $p$ -local 2-periodic complex  $K$ -theory ring spectrum; we will also write  $K_p^\wedge = KU_p^\wedge$  for the  $p$ -adic completion of  $K$ .

It is known from [1–4] that

$$K_0 K = K_0(K) \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(\mathbb{Z}_{(p)}^\times) \subseteq \mathbb{Z}_{(p)}\},$$

and  $K_0 K$  is a free  $\mathbb{Z}_{(p)}$ -module. It follows that

$$K_0^\vee K = \pi_0((K \wedge K)_p^\wedge) = (K_0 K)_p^\wedge.$$

and there is an isomorphism of  $\mathbb{Z}_p$ -adic Banach algebras

$$K_0^\vee K \cong \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

For  $a \in \mathbb{Z}_{(p)}^\times$ , the stable Adams operation

$$\psi^a \in K^0 K \cong \text{Hom}_{\mathbb{Z}_{(p)}}(K_0 K, \mathbb{Z}_{(p)})$$

is determined by the duality pairing

$$\langle \psi^a | f(w) \rangle = f(a).$$

This extends to a continuous duality for  $a \in \mathbb{Z}_p^\times$  on  $f \in \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ ,

$$\langle \psi^a | f \rangle = f(a).$$

For more details on  $K_0(K)$  and  $\text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ , see [10, section 3]; for a broader overview of the connections with  $p$ -adic analysis see [14].

We also recall that  $K_0 K$  is a bicommutative  $\mathbb{Z}_{(p)}$ -Hopf algebra with coproduct  $\Psi$  given by

$$\Psi(f(w)) = f(w \otimes w)$$

and antipode  $\chi$  given by

$$\chi(f(w)) = f(w^{-1}).$$

Using the duality we can obtain a left action of  $K^0 K$  on  $K_0 K$ ; for  $\alpha \in K^0 K$ , we write  $\alpha f(w)$  for this. In particular, if  $a \in \mathbb{Z}_{(p)}^\times$  this coincides with the action of the Adams operation  $\psi^a$ ,

$$\psi^a f(w) = f(a^{-1}w).$$

The reason for the inverse is that we are using the standard left action of the dual of the Hopf algebra  $K_0 K$  defined by

$$\alpha x = \sum_i \langle \alpha(\chi(x'_i)) | x''_i \rangle,$$

where  $\Psi x = \sum_i x'_i \otimes x''_i$ ,  $\Psi(g(w)) = g(w \otimes w)$  and  $\chi(g(w)) = g(w^{-1})$ . In the  $p$ -complete setting, Adams operations indexed on  $\mathbb{Z}_p^\times$  which is the completion of  $\mathbb{Z}_{(p)}^\times$ , so that the assignment

$$\mathbb{Z}_p^\times \times K_r(X)_p^\wedge \rightarrow K_r(X)_p^\wedge; \quad (\alpha, x) \mapsto \psi^\alpha(x)$$

is continuous.

We use notation from [13, chapter IX] and the more recent [11]. For an  $E_\infty$  ring spectrum  $A$  there is a natural power operation  $Q: K_0^\vee(A) \rightarrow K_0^\vee(A)$  (sometimes also called  $\theta$ ) satisfying properties that can be deduced from those listed in [13, theorem IX.3.3] for the homology theories  $K_*(-; p^r)$  with coefficients, and are discussed in [11, section 6], although the version there is for  $\mathbb{Z}/2$ -graded  $K$ -theory, however as we are mainly interested in the case of  $K_*^\vee(K)$  which is concentrated in even degrees, we mostly work with  $K_0^\vee(-)$  but sometimes need to relate this to  $K_{2n}^\vee(-)$  for an integer  $n$ .

The operation is neither additive nor multiplicative, but it satisfies the identities

$$\begin{aligned} Q(x+y) &= Qx + Qy + \frac{1}{p} \left( x^p + y^p - (x+y)^p \right), \\ Q(xy) &= y^p Qx + x^p Qy + p Qx Qy, \end{aligned}$$

or equivalently the operation  $\widehat{Q}$  defined by

$$\widehat{Q}x = p Qx + x^p$$

is additive and multiplicative,

$$\begin{aligned} \widehat{Q}(x+y) &= \widehat{Q}x + \widehat{Q}y, \\ \widehat{Q}(xy) &= \widehat{Q}x \widehat{Q}y. \end{aligned}$$

We also have  $Q1 = 0$ , hence  $\widehat{Q}1 = 1$  and  $\widehat{Q}$  is a (unital) ring homomorphism. Finally, for  $a \in \mathbb{Z}_{(p)}$ ,

$$\begin{aligned} Q(ax) &= a Qx + \frac{(a - a^p)}{p} x^p, \\ \widehat{Q}(ax) &= a \widehat{Q}x. \end{aligned}$$

When  $K_r^\vee(A) = K_r(A)_p^\wedge$ , the operations  $Q$  and  $\widehat{Q}$  are continuous with respect to the  $p$ -adic topology. This allows us to extend these identities to the case where  $\alpha \in \mathbb{Z}_p^\times$ ,

$$\begin{aligned} Q(\alpha x) &= \alpha Qx + \frac{(\alpha - \alpha^p)}{p} x^p, \\ \widehat{Q}(\alpha x) &= \alpha \widehat{Q}x. \end{aligned}$$

Notice that if  $X$  is an infinite loop space and so  $\Sigma_+^\infty X$  is an  $E_\infty$  ring spectrum, then the diagonal map on  $X$  induces a coalgebra structure on  $K_0(\Sigma_+^\infty X)$  (at least if this is  $\mathbb{Z}_{(p)}$ -free) and then  $\widehat{Q}$  is a coalgebra morphism; in particular,  $\widehat{Q}$  preserves coalgebra primitives.

We also mention a useful fact about Adams operations. Let  $\alpha \in \mathbb{Z}_p^\times$  and suppose that  $\psi^\alpha x = \alpha^d x$ . Then

$$\psi^\alpha \widehat{Q}x = \alpha^d \widehat{Q}x,$$

since  $\psi^\alpha$  is a ring homomorphism, hence

$$\begin{aligned} \psi^\alpha \widehat{Q}x &= p Q(\psi^\alpha x) + (\psi^\alpha x)^p \\ &= p Q(\alpha^d x) + (\alpha^d x)^p \\ &= \widehat{Q}(\alpha^d x) = \alpha^d \widehat{Q}x. \end{aligned}$$

## 2. POWER OPERATIONS ON $K_0^\vee K$

We begin with the action of  $Q$  on the basic element  $w \in K_0 K \subseteq K_0^\vee K$ . For  $a \in \mathbb{Z}_{(p)}^\times$ ,

$$\psi^a Q(w) = Q(\psi^a w) = Q(a^{-1} w).$$

Write  $Q(w) = f_0(w)$  where  $f_0 \in \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$  means  $x \mapsto f_0(x)$ , i.e.,  $w$  is the inclusion function  $\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p$ .

By [13, theorem IX.3.3(vi)], for  $k \in \mathbb{Z}$ ,

$$Q(kw) = k Q(w) + \frac{(k - k^p)}{p} w^p,$$

so as  $\mathbb{Z}_{(p)}^\times \subseteq \mathbb{Z}_p^\times$  is dense, this defines a continuous function

$$z f_0(w) + \frac{(z - z^p)}{p} w^p : \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p; \quad (x, y) \mapsto x f_0(y) + \frac{(x - x^p)}{p} y^p.$$

Taking  $y = 1$ , this restricts to the continuous function

$$\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p; \quad x \mapsto x f_0(1) + \frac{(x - x^p)}{p},$$

and as  $f(1) = 0$ , we have

$$f_0(x) = \frac{(x - x^p)}{p}.$$

Hence we have

$$(2.1) \quad Q w = f_0(w) = \frac{(w - w^p)}{p}.$$

For  $n \in \mathbb{N}$ , by [13, theorem IX.3.3(vii)]

$$Q(w^{n+1}) = w^p Q(w^n) + w^{np} Q(w) + p Q(w^n) Q(w)$$

and an easy induction gives the general formula

$$Q(w^n) = \frac{(w^n - w^{np})}{p}.$$

We also have

$$0 = Q(1) = Q(w^n w^{-n}) = w^{np} Q(w^{-n}) + w^{-np} Q(w^n) + p Q(w^n) Q(w^{-n})$$

and so

$$Q(w^{-n}) = \frac{w^{-n} - w^{-np}}{p}.$$

Therefore for  $n \in \mathbb{Z}$ ,

$$(2.2) \quad Q(w^n) = \frac{w^n - w^{np}}{p}.$$

The operation  $\widehat{Q}$  is given by

$$\widehat{Q}(w^n) = \widehat{Q}(w)^n,$$

so for any  $g \in \text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$  we have

$$\widehat{Q}(g(w)) = g(\widehat{Q} w) = g(w),$$

and therefore

$$Q(g(w)) = \frac{1}{p}(g(w) - g(w)^p).$$

This shows that the sequence of polynomial functions defined recursively by  $\theta_0(w) = w$  and for  $n \geq 1$ ,

$$\theta_n(w) = \frac{1}{p}(\theta_{n-1}(w) - \theta_{n-1}(w)^p),$$

is also given by

$$(2.3) \quad \theta_n(w) = Q(\theta_{n-1}(w)).$$

It is known that certain monomials in the  $\theta_n(w)$  form a (topological)  $\mathbb{Z}_p$ -basis for  $K_0^\vee(K)$ , see [3] for example. One interpretation of what we have shown is the following result which seems to have been long known to Mike Hopkins *et al*, but we do not know a published source. We interpret the operation  $Q$  as a realisation of an action of  $\theta$  and therefore  $K_0^\vee K$  becomes a  $p$ -complete  $\mathbb{Z}_p$ - $\theta$ -algebra [11, 12].

**Proposition 2.1.** *The  $p$ -complete  $\mathbb{Z}_p$ - $\theta$ -algebra  $K_0^\vee(K)$  is generated by the element  $w$ . Hence  $K_0^\vee(K)$  is a quotient of the free  $p$ -complete  $\mathbb{Z}_p$ - $\theta$ -algebra  $K_0^\vee(\mathbb{P}S^0)$ , namely*

$$K_0^\vee(K) \cong \mathbb{Z}_p[\theta^s(x) : s \geq 0]_p^\wedge / (((\theta^s(x))^p - \theta^s(x) + p\theta^{s+1}(x) : s \geq 0)).$$

Here the quotient is taken with respect to the  $p$ -adic closure of the ideal generated by the stated elements, indicated by the use of  $((-))$  rather than  $(-)$ . This shows that apart from the  $p$ -adic completion involved,  $K_0^\vee K$  is a colimit of Artin-Schreier extensions of the form

$$\mathbb{Z}_p[X]/(X^p - X + pa)$$

whose mod  $p$  reduction is the étale  $\mathbb{F}_p$ -algebra

$$\mathbb{F}_p[X]/(X^p - X) \cong \prod_{0 \leq r \leq p-1} \mathbb{F}_p.$$

Our discussion also shows that the antipode of  $K_0^\vee(K)$ ,  $\chi$  satisfies

$$(2.4) \quad \chi Q = Q \chi.$$

Suppose that  $A$  is an  $E_\infty$  ring spectrum (or a  $K(1)$ -local  $E_\infty$  ring spectrum). Then we may consider  $K_\bullet^\vee(A)$  where  $K_\bullet^\vee(-)$  denotes the  $\mathbb{Z}/2$ -graded  $p$ -complete theory. The power operation  $Q$  intertwines with the coaction as described in [7, (2.5)], giving

$$(2.5) \quad \Psi Q x = Q(\Psi x)$$

since the antipode  $\chi$  satisfies (2.4) and we have a simpler situation compared to ordinary mod  $p$  homology where the dual Steenrod algebra supports two distinct Dyer-Lashof structures related by the antipode.

### 3. POWER OPERATIONS ON $K_0^\vee KO$ AT 2

In this section we give a brief description of the modification required to describe power operations in  $KO^\vee(KO)$  at the prime  $p = 2$ . For  $KO_*(KO)_{(2)}$ , results of [1, 2] give

- for all  $m \in \mathbb{Z}$ ,  $KO_m(KO)_{(2)} \cong KO_m \otimes KO_0(KO)_{(2)}$ ;
- $KO_0(KO)_{(2)}$  is a countable free  $\mathbb{Z}_{(2)}$ -module;
- $KO_0(KO)_{(2)} = \{f(w) \in \mathbb{Q}[w^2, w^{-2}] : f(\mathbb{Z}_2^\times) \subseteq \mathbb{Z}_2\}$ .

Passing to  $KO_0^\vee(KO)$  and recalling that the squaring homomorphism

$$\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2) \rightarrow 1 + 8\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$$

is surjective, we see that the natural  $E_\infty$  morphism  $KO \rightarrow KU$  induces a monomorphism of 2-complete  $\theta$ -algebras  $KO_0^\vee(KO) \rightarrow K_0^\vee(K)$  coinciding with the inclusion of the continuous functions factoring through  $(-)^2$ .

It is clear that  $Q$  restricts to  $KO_0^\vee(KO)$  and is given by

$$Q(f) = \frac{(f - f^2)}{2}.$$

The following inductively defined elements provide a topological basis for  $KO_0^\vee(KO)$ :

$$\Theta_0 = \frac{1 - w}{2}, \quad \Theta_n = \frac{\Theta_{n-1} - \Theta_{n-1}^2}{2} \quad (n \geq 1).$$

Then the distinct monomials  $\Theta_0^{\varepsilon_0} \Theta_1^{\varepsilon_1} \cdots \Theta_\ell^{\varepsilon_\ell}$  with  $\varepsilon_j = 0, 1$  form a topological basis.

#### 4. THE COMPLETED $K$ -THEORY OF FREE ALGEBRAS

In this section we will describe  $K_0^\vee(\mathbb{P}X)$ , at least for spectra  $X$  for which  $K_0^\vee(X)$  is suitably restricted. For our purposes, it will suffice to assume that  $X$  is a CW spectrum with only finitely many even dimensional cells. It will be useful to examine how  $K_0^\vee(\mathbb{P}X)$  behaves for such complexes.

Suppose that the  $(n-1)$ -skeleton  $X^{[n-1]}$  of  $X$  is defined. Then the  $n$ -skeleton  $X^{[n]}$  is a pushout defined by a diagram of the form

$$\begin{array}{ccc} \bigvee_i S^{n-1} & \longrightarrow & \bigvee_i D^n \\ \downarrow & \lrcorner & \downarrow \\ X^{[n-1]} & \longrightarrow & X^{[n]} \end{array}$$

for a finite wedge of spheres  $\bigvee_i S^{n-1}$ . Similarly there is a pushout diagram of commutative  $S$ -algebras

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^{n-1}) & \longrightarrow & \mathbb{P}(\bigvee_i D^n) \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{P}(X^{[n-1]}) & \longrightarrow & \mathbb{P}(X^{[n]}) \end{array}$$

so  $(\mathbb{P}X)^{\langle n \rangle} = \mathbb{P}(X^{[n]})$  is the  $E_\infty$   $n$ -skeleton of the CW commutative  $S$ -algebra  $\mathbb{P}X$ .

If the cells of  $X$  are all even dimensional, we only encounter pushout diagrams of the form

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^{2m-1}) & \longrightarrow & \mathbb{P}(\bigvee_i D^{2m}) \\ \downarrow & \lrcorner & \downarrow \\ (\mathbb{P}X)^{\langle 2m-2 \rangle} & \longrightarrow & (\mathbb{P}X)^{\langle 2m \rangle} \end{array}$$

where

$$(\mathbb{P}X)^{\langle 2m \rangle} \cong (\mathbb{P}X)^{\langle 2m-2 \rangle} \wedge_{\mathbb{P}(\bigvee_i S^{2m-1})} \mathbb{P}(\bigvee_i D^{2m}).$$

To calculate  $K_*^\vee((\mathbb{P}X)^{\langle 2m \rangle})$  we may use a Künneth spectral sequence of the form

$$(4.1) \quad E_{s,t}^2 = \mathrm{Tor}_{s,t}^{K_*^\vee(\mathbb{P}(\bigvee_i S^{2m-1}))}(K_*^\vee((\mathbb{P}X)^{\langle 2m-2 \rangle}), K_*) \implies K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m \rangle}),$$

where the internal  $t$  grading is in  $\mathbb{Z}/2$ , i.e., it is an integer mod  $p$ . This is essentially described in [16], but we will require its multiplicativity, and also the fact that it inherits an action of power operations. The latter structure is constructed in a similar fashion to the mod  $p$  Dyer-Lashof operations in [20].

**Proposition 4.1.** *The spectral sequence (4.1) collapses at  $E^2$  to give*

$$K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m \rangle}) = K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m-2 \rangle})[Q^s x_i : s \geq 0, i]_{\widehat{p}},$$

where each  $x_i$  is in even degree.

*Proof.* Recall from [11] that

$$K_*^\vee\left(\mathbb{P}\left(\bigvee_i S^{2m-1}\right)\right) = \Lambda(z_i)_{\widehat{p}},$$

the  $p$ -completed exterior algebra on odd degree generators  $z_i \in K_1^\vee(\mathbb{P}(\bigvee_i S^{2m-1}))$ , each of which originates on a wedge summand.

The  $E^2$ -term is a divided power algebra over  $K_*^\vee((\mathbb{P}X)^{\langle 2m-2 \rangle})$  on generators of bidegree  $(1, 1)$ , each represented in the cobar complex by  $[Q^s z_i]$ . We will write  $\gamma_r([Q^s z_i])$  for the  $r$ -th divided power of this element and recall that the particular elements  $\gamma_{(r)}([Q^s z_i]) = \gamma_{p^r}([Q^s z_i])$  generate the algebra subject to relations of the form

$$\gamma_{(r)}([Q^s z_i])^p = \binom{p^{r+1}}{p^r, \dots, p^r} \gamma_{(r+1)}([Q^s z_i]),$$

where the multinomial coefficient satisfies

$$\binom{p^{r+1}}{p^r, \dots, p^r} = pt$$

for some integer  $t$  not divisible by  $p$ . For degree reasons there can only be trivial differentials, so the only issue still to be resolved is that of the multiplicative structure.

We follow a similar line of argument to that of [20]. In the spectral sequence we have

$$Q[z_i] = [Q^s z_i],$$

so it remains to relate this element to a  $p$ -th power in the target.

By [13, chapter IX, theorem 3.3(viii)], if  $Z_i$  is represented by  $[z_i]$ , then  $\widehat{Q} Z_i = Z_i^p + p Q Z_i$  is represented by  $[Q z_i]$ . It follows that each such  $Z_i$  has non-trivial  $p$ -th power also represented in the 1-line. This can be extended to show that each  $\gamma_{(r)}([Q^s z_i])$  represents an element with non-trivial  $p$ -th power. Finally, an easy argument shows that the target is a completed polynomial algebra as stated.  $\square$

It is also useful to generalise this to the case of a CW spectrum  $Y$  with chosen 0-cell  $S^0 \rightarrow Y$ , where  $S^0 \xrightarrow{\sim} S$  is the functorial cofibrant replacement of  $S$  in the model category of  $S$ -modules. We may then consider the reduced free commutative  $S$ -algebras  $\widetilde{\mathbb{P}}Y$  which is defined as the homotopy pushout of the diagram of solid arrows

$$\begin{array}{ccc} \mathbb{P}S^0 & \longrightarrow & \mathbb{P}Y \\ \downarrow & \lrcorner & \downarrow \\ S & \dashrightarrow & \widetilde{\mathbb{P}}Y \end{array}$$

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where the vertical map is the canonical multiplicative extension of  $S^0 \rightarrow S$ ; see [6] for more on this construction. As a particular case, we can consider a map  $f: S^{2m-1} \rightarrow S^0$  and form its mapping cone  $C_f = S^0 \cup_f D^{2m}$ . Then take  $S//f = \mathbb{P}C_f$  to be a homotopy pushout for the diagram

$$\begin{array}{ccc} \mathbb{P}S^0 & \longrightarrow & \mathbb{P}C_f \\ \downarrow & \lrcorner & \downarrow \\ S & \dashrightarrow & S//f \end{array}$$

and there is an associated Künneth spectral sequence

$$(4.2) \quad E_{s,t}^2 = \text{Tor}^{K_*^\vee(\mathbb{P}S^0)}(K_*, K_*^\vee(\mathbb{P}C_f)) \implies K_{s+t}^\vee(S//f).$$

It is easily seen that

$$K_*^\vee(\mathbb{P}S^0) = \mathbb{Z}_p[\mathbb{Q}^s x_0 : s \geq 0]_p^\wedge$$

is a subalgebra of

$$K_*^\vee(\mathbb{P}C_f) = \mathbb{Z}_p[\mathbb{Q}^s x_0, \mathbb{Q}^s x_{2m} : s \geq 0]_p^\wedge,$$

and the spectral sequence has

$$E_{0,*}^2 = K_* \otimes_{K_*^\vee(\mathbb{P}S^0)} K_*^\vee(\mathbb{P}C_f) = \mathbb{Z}_p[\mathbb{Q}^s x_{2m} : s \geq 0]_p^\wedge, \quad E_{r,*}^2 = 0 \quad (r \geq 1).$$

It follows that

$$K_*^\vee(S//f) = \mathbb{Z}_p[\mathbb{Q}^s x_{2m} : s \geq 0]_p^\wedge.$$

Provided we know the coaction for  $K_*^\vee(C_f)$ , that for  $K_*^\vee(S//f)$  follows formally. In general we have only the following possible kind of coaction,

$$\Psi(x_{2m}) = w^n \otimes x_{2m} + c(f)(1 - w^m),$$

where  $c(f)$  is a certain kind of rational number. Then

$$\Psi(\mathbb{Q}^s x_{2m}) = \mathbb{Q}^s(\Psi x_{2m})$$

which involves iterated application of  $\mathbb{Q}$ .

## 5. SOME EXAMPLES BASED ON ELEMENTS OF HOPF INVARIANT 1

We will consider the examples  $S//\eta$ ,  $S//\nu$  and  $S//\sigma$  at the prime  $p = 2$  discussed in [8]. Similar considerations apply to examples constructed using elements in the image of the  $J$ -homomorphism at an arbitrary prime. In order to study such an example, it is necessary to determine the  $K_0^\vee(K)$ -coaction on  $K_0^\vee(S//f)$ .

We will use the following elements of  $K_0^\vee(K)$ :

$$\Theta_0 = \frac{(1-w)}{2}, \quad \Theta_n = \frac{(\Theta_{n-1} - \Theta_{n-1}^2)}{2} \quad (n \geq 1).$$

By results of [3],  $K_0^\vee(K)$  has a topological basis consisting of the monomials

$$\Theta_0^{\varepsilon_0} \Theta_1^{\varepsilon_1} \cdots \Theta_\ell^{\varepsilon_\ell} \quad (\varepsilon_i = 0, 1).$$

If we view these as continuous functions on  $\mathbb{Z}_2^\times$ , then for a 2-adic unit  $\alpha$  expressed as

$$\alpha = 1 - (2a_0 + 2^2a_1 + \cdots + 2^{r+1}a_r + \cdots)$$

with  $a_r = 0, 1$ , in  $\mathbb{Z}_2$  we have

$$\Theta_r(\alpha) \equiv a_r \pmod{2}.$$



We also know that  $Q\Theta_s = \Theta_{s+1}$ , hence  $\Theta_s = Q^s\Theta_0$ .

In the case where  $f = \eta$ , we can take the generator  $x_2$  to have coaction

$$\Psi(x_2) = \frac{(1-w)}{2} \otimes 1 + w \otimes x_2 = \frac{(1-w)}{2} + wx_2 = \Theta_0 + wx_2,$$

where we suppress the tensor product symbols when the meaning seems clear without them. We also have for the coproduct in  $K_0^\vee(K)$ ,

$$\Psi\Theta_0 = \Theta_0 \otimes 1 + w \otimes \Theta_0,$$

and also

$$\Psi Qx_2 = w Qx_2 + w\Theta_0x_2^2 - w\Theta_0x_2 + \Theta_1.$$

Without further calculation we see that there is a homomorphism of topological comodule algebras

$$\mathbb{Z}_2[x_2]_2^\wedge \rightarrow K_0^\vee(K); \quad x_2 \mapsto \Theta_0.$$

This is induced from a morphism of  $E_\infty$  ring spectra  $S//\eta \rightarrow K$  arising from the fact that the composition of  $\eta: S^1 \rightarrow S$  with the unit  $S \rightarrow K$  is null homotopic. Therefore there is an extension to a continuous epimorphism

$$K_0^\vee(S//\eta) \rightarrow K_0^\vee(K); \quad Q^s x_2 \mapsto \Theta_s.$$

This displays  $K_0^\vee(K)$  as a quotient of the free  $\theta$ -algebra  $K_0^\vee(S//\eta)$  as in Proposition 2.1.

**Theorem 5.1.** *There is a  $K(1)$ -local equivalence*

$$S//\eta \xrightarrow{\sim} \bigvee_{j \geq 0} K.$$

We will give a proof of this in a future paper. Notice that there is an  $E_\infty$  morphism  $S//\eta \rightarrow kU$  and this induces a surjection on  $\pi_*(-)$  but not on  $H_*(-; \mathbb{F}_2)$ . Hence  $kU$  cannot be a retract of  $S//\eta$  2-locally or after 2-completion. However, multiplication by the Bott map induces a cofibre sequence

$$\Sigma^2 kU \rightarrow kU \rightarrow H\mathbb{Z}$$

where  $KU \wedge H\mathbb{Z}$  is rational. Therefore  $\Sigma^2 kU \rightarrow kU$  is a  $K(1)$ -local equivalence, so it induces an isomorphism on  $K^\vee(-)$ .

Notice that

$$w^2 = (1 - 2\Theta_0)^2 = 1 - 4(\Theta_0 - \Theta_0^2) = 1 - 8\Theta_1,$$

so

$$1 - w^2 = 8\Theta_1.$$

Similarly,

$$w^4 = 1 - 16(\Theta_1 - \Theta_1^2) + 48\Theta_1^2,$$

and therefore

$$1 - w^4 = 16(\Theta_1 - \Theta_1^2) - 48\Theta_1^2 = 32\Theta_2 - 48\Theta_1^2.$$

Such identities allow us to describe the groups

$$\mathrm{Ext}_{K_*(K)}^{1,2n}(K_*, K_*) = \mathrm{Pr} K_{2n}(K)/(\eta_L - \eta_R)K_{2n}$$

that detect the 2-primary part of image of the  $J$ -homomorphism through the  $e$ -invariant. Here  $\mathrm{Pr}$  denotes the subgroup of primitive elements which satisfy

$$\Psi(x) = 1 \otimes x + x \otimes 1,$$

and  $\eta_L, \eta_R$  denote the left and right units respectively. When  $n = 1, 2, 4$ , these groups are cyclic with the following orders and generators:

- 2, generator represented by  $u\Theta_0$ ;
- 8, generator represented by  $u^2\Theta_1$ ;
- 16, generator represented by  $u^4(2\Theta_2 - 3\Theta_1^2)$ .

Here we write  $u \in K_2$  for the Bott generator. In the first and last cases, a generator of  $(\text{im } J)_{2n-1}$  maps to the generator, but in the middle case only the multiples of  $2u^2\Theta_1$  are hit; for details see [21, 23].

For

$$K_0^\vee(S//\nu) = \mathbb{Z}_2[\mathbb{Q}^s x_4 : s \geq 0]_2^\wedge, \quad K_0^\vee(S//\sigma) = \mathbb{Z}_2[\mathbb{Q}^s x_8 : s \geq 0]_2^\wedge,$$

we have the coactions

$$\Psi x_4 = w^2 \otimes x_4 + 2\Theta_1, \quad \Psi x_8 = w^4 \otimes x_8 + 2\Theta_2 - 3\Theta_1^2.$$

Finally, we note that there is an  $E_\infty$  morphism  $S//\nu \rightarrow kO$  inducing an epimorphism on  $\pi_*(-)$  which is not an epimorphism on  $H_*(-; \mathbb{F}_2)$ . The composition  $S//\nu \rightarrow kO \rightarrow KO$  induces a  $K(1)$ -local splitting.

**Theorem 5.2.** *There is a  $K(1)$ -local equivalence*

$$S//\nu \xrightarrow{\sim} \bigvee_{j \geq 0} \Sigma^{4\rho(j)} KO,$$

for some numerical function  $\rho$ .

## APPENDIX A. COACTIONS AND CONTINUOUS ACTIONS

In this appendix we recall the relationship between  $L$ -complete  $K_*^\vee(K)$ -comodules and  $L$ -complete modules with continuous  $\mathbb{Z}_p^\times$  linear actions. We are working in the context where  $K_*^\vee(X)$  is a  $\mathbb{Z}/2$ -graded  $L$ -complete module for the pair  $(\mathbb{Z}_p, (p))$ . Here the category of  $L$ -complete modules  $\mathcal{M} = \mathcal{M}_{\mathbb{Z}_p}$  behaves better than its higher chromatic analogues. For example, since  $\dim \mathbb{Z}_p = 1$ ,

$$L_1\left(\prod_{\alpha} M_{\alpha}\right) = 0$$

and it follows that pro-free modules are  $L$ -flat as defined in [5]. In particular, for a pro-free  $L$ -complete Hopf algebroid  $(\mathbb{Z}_p, \Gamma)$  as defined in [5, definition 2.4], the category of left (or right)  $\Gamma$ -comodules has kernels so we can do homological algebra therein. We remark also that by [5, lemma 1.5], every  $L$ -complete module is Hausdorff with respect to its  $p$ -adic topology. Notice also that our Hopf algebroids are actually Hopf algebras since the left and right  $p$ -adic completions of  $K_0(K)$  coincide and the difference between left and right units in  $K_0(K)$  is divisible.

On the other hand, when  $G$  is a profinite group, a  $p$ -adically continuous left action of  $G$  on an  $L$ -complete module  $M$  gives rise to a left  $\Gamma(G)$ -comodule structure for the pro-free  $L$ -complete Hopf algebroid  $(\mathbb{Z}_p, \Gamma(G))$  where

$$\Gamma(G) = \text{Map}^c(G, \mathbb{Z}_p).$$

Here

$$\Psi: M \rightarrow \Gamma(G) \hat{\otimes} M \cong \text{Map}^c(G, M); \quad \Psi(x)(\gamma) = \gamma^{-1}x.$$

The converse construction also works for all  $L$ -complete  $\Gamma(G)$ -comodules: for an  $L$ -complete  $\Gamma(G)$ -comodule structure  $\Psi: M \rightarrow \Gamma(G) \hat{\otimes} M \cong \text{Map}^c(G, M)$  there is a continuous  $G$ -action given

$$G \times M \rightarrow M; \quad \gamma \cdot m = \Psi(\gamma^{-1}).$$

Thus  $L$ -complete  $\Gamma(G)$ -comodules and continuous  $L$ -complete  $G$ -modules are essentially the same thing. More precisely, there is an isomorphism of abelian categories

$$\mathcal{M}(G) \cong \mathcal{M}(\Gamma(G))$$

Now we will discuss cohomology for these structures. We refer to [15, 22] for background on this version of the Adams-Novikov spectral sequence adapted to the  $K(1)$ -local context.

The left exact additive functors

$$\begin{aligned} \mathcal{M}(G) &\rightarrow \mathcal{M}; \quad m \mapsto M^G, \\ \mathcal{M}(\Gamma(G)) &\rightarrow \mathcal{M}; \quad m \mapsto \mathbb{Z}_p \hat{\square}_{\Gamma(G)} M, \end{aligned}$$

where  $\mathbb{Z}_{(p)} \hat{\square}_{\Gamma(G)} M$  denotes the completed cotensor product which is the equaliser of

$$\mathbb{Z}_p \hat{\square}_{\Gamma(G)} M \longrightarrow M \xrightleftharpoons[1 \otimes \text{Id}]{\Psi} \Gamma(G) \hat{\otimes} M$$

viewed as a diagram in  $\mathcal{M}$ . These functors clearly agree, so their left derived functors  $\mathcal{H}_c^*(G; -)$  and  $\widehat{\text{Cotor}}_{\Gamma(G)}^*(\mathbb{Z}_p, -)$  also agree.

This means that we might as well work with continuous cohomology and make use of the descent spectral sequence discussed for example in [15, 22]. We are working in the case where the chromatic height is 1 and the Morava stabilizer group is

$$\mathbb{Z}_p^\times = \begin{cases} \{\pm 1\} \times (1 + 4\mathbb{Z}_2) & \text{if } p = 2, \\ C_{p-1} \times (1 + p\mathbb{Z}_p) & \text{if } p \text{ is odd.} \end{cases}$$

The open normal subgroups are of the form  $A \times (1 + p^r \mathbb{Z}_p)$ , where

- $A \leq \{\pm 1\}$  and  $r \geq 2$  if  $p = 2$ ,
- $A \leq C_{p-1}$  and  $r \geq 1$  if  $p$  is odd.

Now from [15] we have the *Descent Spectral Sequence*.

**Theorem A.1.** *For a spectrum  $X$  there is a conditionally convergent spectral sequence*

$$E_2^{s,t} = \mathcal{H}^s \left( \mathbb{Z}_p^\times; \pi_t \left( \text{hocolim}_{N \triangleleft \mathbb{Z}_p^\times} (L_{K(1)}(K \wedge X))^{hN} \right) \right) \implies \pi_{t-s}(L_{K(1)}X),$$

where the homotopy colimit is taken over all finite index normal subgroups and the homotopy groups have a discrete action of  $\mathbb{Z}_p^\times$ .

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